

ANOMALY INDUCED EFFECTIVE ACTION COMBINED WITH GENERALIZED BRANS-DICKE THEORY

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Abstract

We consider the anomaly induced effective action in $\mathcal{N}=4$ super Yang-Mills theory in interaction with the Brans-Dicke (BD) field. The generalization of the BD theory so as to permit an energy exchange between the scalar field and ordinary matter fields, was recently worked out by T. Clifton and J. D. Barrow [Phys. Rev. D **73**, 104022 (2006)]. We derive the scalar field equations for the dilaton field, and the BD field, and discuss the Friedmann equation in the general case. The present paper is a continuation of an investigation some years ago dealing with the case of conformal anomaly plus ordinary classical gravity [I. Brevik and S. D. Odintsov, Phys. Lett. B **455**, 104 (1999)].

Keywords: Quantum cosmology; effective action; Brans-Dicke theory

1 Introduction

The recent paper of Clifton and Barrow [1] sheds light on an interesting possibility in the formulation of scalar-tensor gravity theories, namely to allow for an *exchange* of energy (and momentum) between the scalar field ϕ and the ordinary matter fields. This kind of generalization implies an enrichment of the formalism; as these authors point out, one may even obtain variations in the gravitational "constant" G . the field-matter exchange means that the energy-momentum tensor $T_M^{\mu\nu}$ of matter is no longer divergence-free,

$$\nabla_\nu T_M^{\mu\nu} = f^\mu, \quad (1)$$

where f^μ is the force density from the field on the matter. What we shall focus attention in the following, is the Brans-Dicke theory [2].

First of all we note, however, that this separation into two interacting subsystems - fields and matter - bears a striking resemblance to the theory of the electromagnetic field in continuous media. Actually, this is the key element in the famous Abraham-Minkowski controversy, a problem that has been discussed with more or less intensity since Abraham and Minkowski formulated their energy-momentum expressions around 1910. The advent of accurate experiments, in particular, has helped us to get more insight into this complicated field-matter interacting system. Some years ago, the present author made a review of the experimental status in the field [3]. There is by now a rather big literature in this field; some papers are listed in Ref. [4, 5, 6, 7, 8, 9, 10, 11].

What we shall consider in the following is however a different theme, namely to what extent the inclusion of the *conformal anomaly* for scalar fields influences the equations of the interacting Brans-Dicke theory. We shall consider quantum $\mathcal{N} = 4$ super Yang-Mills theory interacting in covariant way with $\mathcal{N} = 4$ conformal supergravity. The induced large N effective action for such a theory can be calculated on a non-supersymmetric dilatonic-gravitational background using the conformal anomaly found via the AdS/CFT correspondence. Our intention is to show this calculation in the Brans-Dicke background. On a purely bosonic background with only non-zero gravitational and dilaton fields, the conformal anomaly for $\mathcal{N} = 4$ super YM theory was calculated in [12] via the AdS/CFT correspondence [13, 14, 15]. Our calculation is a generalization of that presented earlier in Ref. [16].

We mention that four-dimensional quantum cosmology with account taken of dilaton-dependent conformal anomaly was considered in Refs. [17, 18, 19].

We consider only the quantum effects of the $\mathcal{N} = 4$ super Yang-Mills theory together with the Brans-Dicke theory as a specific example. We might in principle consider an arbitrary conformal quantum field coupled with a dilaton different from the Brans-Dicke scalar. This would amount to different numerical values for the coefficients of conformal anomaly, and in some cases it would lead to the appearance of new, dilaton dependent terms [19].

In the next section we establish the field equations for the dilaton field χ and the Brans-Dicke field ϕ , and find the Friedmann equation under specified simplifying conditions. In section 3, some typical properties of the solutions of the Friedmann equation are discussed.

2 Basic formalism

Let us start from the anomaly induced effective action [20]. We write it in a non-covariant local form, and we limit ourselves to a conformally flat metric, i.e., $g_{\mu\nu} = e^{2\sigma}\eta_{\mu\nu}$, where $\eta_{\mu\nu}$ is the Minkowski metric. We assume also that only the real part of the dilaton field χ is different from zero.

The anomaly induced effective action takes under these conditions a relatively simple form:

$$W = - \int d^4x \left\{ 2b' \sigma \square^2 \sigma - 3[b'' + \frac{2}{3}(b + b')] \right. \\ \left. \times (\square \sigma + \partial_\mu \sigma \partial^\mu \sigma)^2 + C \sigma \chi \square^2 \chi \right\}. \quad (2)$$

Here $\square = \partial^\mu \partial_\mu$, all derivatives being flat ones. The various constants are given as

$$b = \frac{N^2 - 1}{(4\pi)^2} \frac{N_s + 6N_f + 12N_v}{120}, \quad (3)$$

$$b' = -\frac{N^2 - 1}{(4\pi)^2} \frac{N_s + 11N_f + 62N_v}{360}, \quad (4)$$

$$C = \frac{N^2 - 1}{(4\pi)^2} N_v. \quad (5)$$

In the $\mathcal{N} = 4$ SU(N) super YM theory, the scalar, fermion and vector numbers are $N_s = 6$, $N_f = 2$, $N_v = 1$, giving

$$b = -b' = \frac{C}{4} = \frac{N^2 - 1}{4(4\pi)^2}. \quad (6)$$

We assume that the scale factor $a(\eta)$ depends only on conformal time, $\sigma(\eta) = \ln a(\eta)$. Also, χ is assumed to depend only on η . One now has to add to the action (2) the contribution from the Brans-Dicke field, and from matter:

$$S_{BD} = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left(\phi R - \frac{\omega}{\phi} \partial_\mu \phi \partial^\mu \phi + 16\pi L_M \right). \quad (7)$$

Here R is the curvature scalar, ω is the Brans-Dicke coupling constant, and L_M is the Lagrangian density of the matter fields. If $\omega \rightarrow \infty$ this theory reduces to general relativity. Actually, by putting $1/\omega = G$ we recover

straightaway the action integral for classical general relativity

$$S_{cl} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} (R + 16\pi G L_M). \quad (8)$$

(Note that in Ref. [16], W and S_{cl} were defined with the opposite sign; our present convention is made to agree with the standard convention for the Brans-Dicke term (7).)

There are thus two scalar fields present: the field χ coming from the conformal anomaly, and the Brans-Dicke field ϕ . Note that χ is defined to be dimensionless, while the dimension of ϕ is cm^{-2} . These fields may be pictured as two "fluids". Also, the matter fields may be associated with a material fluid.

Consider first the field equation for χ . Since the Brans-Dicke action does not depend on χ we get the same equation as in the conformal anomaly-gravity case [16]:

$$\ln a \chi'''' + (\ln a \chi)''' = 0, \quad (9)$$

prime meaning differentiation with respect to η . We can here make a transformation to cosmological time t via the relation $dt = a(\eta)d\eta$. The result is rather complicated,

$$\begin{aligned} & 2a \ln a Y[\chi, a] + \chi a^3 \ddot{a} a + 4a^3 \dot{a} \ddot{a} \chi + 3a^2 \chi \dot{a} \ddot{a} \\ & + 4a^3 \dot{\chi} \ddot{a} + 6a^3 \ddot{\chi} \dot{a} + 12a^2 \ddot{\chi} \ddot{a} + 12a^2 \ddot{\chi} \dot{a}^2 + 14a^2 \dot{\chi} \dot{a} \ddot{a} \\ & + a \chi \ddot{a} \dot{a}^2 + a^2 \chi \ddot{a}^2 + 4a \dot{\chi} \dot{a}^3 = 0, \end{aligned} \quad (10)$$

where

$$\begin{aligned} Y[\chi, a] &= a^3 \ddot{a} \chi + 6a^2 \dot{a} \ddot{a} \chi + 4a^2 \ddot{a} \ddot{\chi} + 7a \dot{a}^2 \ddot{\chi} \\ &+ 4a \dot{a} \ddot{a} \dot{\chi} + a^2 \ddot{a} a \dot{\chi} + \dot{a}^3 \dot{\chi}. \end{aligned} \quad (11)$$

We shall consider only approximate solutions of Eq. (10) when the term with $\ln a$ can be dropped.

The field equation for ϕ can be found analogously. Since W does not depend on ϕ we have to vary the expression (7) only. We assume that the material fluid has energy density ρ and pressure p , and assume that the equation of state can be written in the conventional form

$$p = (\gamma - 1)\rho, \quad (12)$$

where γ is a constant, the case $\gamma = 2$ (Zeldovich fluid) denoting the upper limit for γ . The field equation for ϕ becomes the same as in the pure Brans-Dicke case [1]:

$$\ddot{\phi} + 3H\dot{\phi} = \alpha(4 - 3\gamma)\rho - 2\alpha\frac{\dot{\phi}}{\phi}f^0, \quad (13)$$

where

$$\alpha = \frac{8\pi}{3 + 2\omega}. \quad (14)$$

Finally there is the Friedmann equation, obtained by varying the total action $W + S_{BD}$ with respect to the metric tensor. In order to simplify the formalism somewhat, we exploit that the coefficient b'' is known to be ambiguous. We have a freedom in choosing its value. As advocated in Ref. [21], one can well take

$$3b'' + 2b = 0. \quad (15)$$

(The physical advantage of this choice is that no $\square R$ -term remains in the conformal anomaly.) As for the conformal anomaly part, considerable simplification of the Friedmann equation thereby occurs. In the pure conformal anomaly case, we can write the square of the Hubble parameter as [16]

$$H^2 \Big|_{CA} = -\frac{1}{24} \frac{C\chi Y[\chi, a]}{b' \ddot{a}a^2}. \quad (16)$$

(Here the contribution from the R -term S_d in the action, Eq. (6) in [16], has been omitted.) In the Brans-Dicke case [1]

$$H^2 \Big|_{BD} = \frac{8\pi}{3\phi}\rho + \frac{\omega}{6}\frac{\dot{\phi}^2}{\phi^2} - H\frac{\dot{\phi}}{\phi}, \quad (17)$$

leading to the conventional relation $H^2 = (8\pi G/3)\rho$ in the GR case ($\phi = 1/G$). Now combining expressions (16) and (17) we may construct the Hubble parameter in the general case as

$$H^2 = -\frac{1}{24} \frac{C\chi Y[\chi, a]}{b' \ddot{a}a^2} + \frac{8\pi}{3\phi}\rho + \frac{\omega}{6}\frac{\dot{\phi}^2}{\phi^2} - H\frac{\dot{\phi}}{\phi}. \quad (18)$$

Obviously, this expression agrees with known expressions in the two limiting cases.

3 Remarks on the solutions

We will not discuss in detail the solutions of the complicated equation (18), but will note some general properties that can be verified almost by a mere inspection of the equation.

First of all, one may ask at which stage in the history of the universe the influence from the first term in Eq. (18) was important. In order to investigate this point, we may replace the Brans-Dicke theory with general relativity again and estimate the ratio between the magnitudes of the conformal anomaly term and the classical term, $(8\pi/3\phi)\rho \rightarrow (8\pi G/3)\rho$. Assuming radiation dominance in the very early universe, meaning in standard notation that $\rho = \rho_0(a_0/a)^4$, we have

$$a = \left(\frac{32\pi G}{3} \rho_0 a_0^4 \right)^{1/4} t^{1/2}, \quad (19)$$

leading to $\rho = 3/(32\pi G t^2)$. From Eq. (6), it follows that $-b' \sim C$. As for the term $Y[\chi, a]$, we let its magnitude be represented by the first term on the right in Eq. (11). Making use of Eq. (19) we then obtain after a brief calculation that the ratio between the first and the second term to the right in Eq. (18) becomes of order $a\chi^2$, thus quite simple. Now, on physical grounds we would expect χ to decay exponentially with time. Let us express this as $\chi \propto e^{-\beta \tilde{H}t}$, to conform with the notation in Eq. (20) below. The expression $a\chi^2$ thus approaches zero when $t \rightarrow 0$. At the beginning of the universe, the relative importance of conformal anomaly was equal to zero. The influence of conformal anomaly decreases to zero for large times also, like $t^{-1/2}e^{-2\beta \tilde{H}t}$. The maximum contribution occurs at an intermediate time, $t = 1/(2\beta \tilde{H})$.

We now consider two special examples.

3.1 The case of conformal anomaly-general relativity

This case was discussed in an earlier paper [16]. For completeness, we summarize the main results also here.

We look for approximate solutions based upon the de Sitter forms:

$$a(t) = \tilde{a}_0 e^{\tilde{H}t}, \quad \chi(t) = \chi_0 e^{-\beta \tilde{H}t}, \quad (20)$$

where \tilde{H} and β are constants. We consider only very early times, for which $\tilde{H}t \ll 1$. It implies, as mentioned above, that the $\ln a$ term in Eq. (10) can

be dropped. From the remaining part of Eq. (10) we obtain three different modes, corresponding to $\beta = \{3/2, 2.62, 0.38\}$. For each mode we find

$$\tilde{H}^2 = -\frac{1}{16\pi G} \left[b' + \frac{C}{24} \chi_0^2 (\beta^4 - 6\beta^3 + 11\beta^2 - 6\beta) \right]^{-1}. \quad (21)$$

For the modes $\beta = 2.62$ or $\beta = 0.38$, we calculate \tilde{H} to be less than the Hubble parameter in the non-dilatonic case. The effect of the dilaton is thus to slow down the inflation.

For the mode $\beta = 3/2$ the situation is reversed: the value of \tilde{H} is larger than in the non-dilatonic case and the dilaton acts to speed up the inflation.

3.2 The case $C = 0$

We now assume the general conformal anomaly - Brans Dicke case, but impose the condition that the coefficient C is equal to zero. According to Eq. (5) this means that $N_v = 0$. The vector contribution to the super YM theory is thus omitted.

The effect that this simple condition has on the formalism is dramatic: in the Friedmann equation (18) there is no influence from the conformal anomaly at all. The Hubble parameter evolves in time as if the Brans-Dicke field were the only scalar field. This does not imply that the dilaton field is equal to zero, however. Still, χ evolves according to the field equations (9) or (10).

Similarly, the condition $C = 0$ is seen to be important also in the expression (21). We now get simply $\tilde{H}^2 = -1/(16\pi G b')$, so that there is no dilatonic influence on the Hubble parameter.

4 Summary

The starting point for our analysis of the situation where there is both a conformal anomaly dilaton field χ and a Brans-Dicke field ϕ present, was to combine the actions in Eqs. (2) and (7). We found the field equation for χ to be given by Eq. (9) (in terms of conformal time η), or by Eq. (10) (in terms of cosmological time t). The field equation for ϕ is given by Eq. (13). This equation is written in a form general enough to permit interaction between ϕ and the matter fields [1].

A main point in the analysis was to construct the expression for H^2 in Eq. (18) as the sum of the expressions (16) and (17) for the two separate subsystems. This form for the Friedmann equation is based upon the choice of the gauge condition (15), which is a permissible choice because of the ambiguity of the coefficient b'' . We found the relative importance of the conformal anomaly to go to zero both for $t \rightarrow 0$ and for large values of t , and to have a maximum at some intermediate time.

The special case $C = 0$ is of interest. It corresponds to omitting the vector contribution ($N_v = 0$) to the super YM theory. Then, the influence from conformal anomaly on the Hubble parameter becomes simply zero. We get the same solutions for the Brans-Dicke field as if there were no conformal anomaly at all, even under the very general conditions studied in [1].

In general, we have discussed the situation where the scalar field from the conformal anomaly is different from the scalar field in the Brans-Dicke theory (i.e., $\mathcal{N} = 4$ SYM is considered in the Brans-Dicke background). However, one can also identify the dilaton from conformal anomaly with the Brans-Dicke scalar, as was done in Ref. [22]. Moreover, other backgrounds can also be considered, like the AdS background [23].

Finally, it is of interest to note that our problem may easily be generalized to the case of a brane-world scenario with a dilaton dependent conformal anomaly and inflation [24, 25, 26].

Acknowledgment

I thank Sergei D. Odintsov for his many valuable comments on the manuscript.

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